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NUMERICAL ANALYSIS OF A FRICTIONAL CONTACT PROBLEM FOR THERMO-ELECTRO-ELASTIC MATERIALS

Youssef Ouafik
National School of Applied Sciences of Safi, Cadi Ayyad University, Safi, Morocco
e-mail: y.ouafik@uca.ma

A numerical method is presented for a mathematical model which describes the frictional contact between a thermo-electro-elastic body and a conductive foundation. The contact is described by Signorini's conditions and Tresca's friction law including electrical and thermal conductivity conditions. Our aim is to present a detailed description of the numerical modelling of the problem. To this end, we introduce a discrete scheme based on the finite element method. Under some regularity assumptions imposed on the true solution, optimal order error estimates are derived for the linear element solution. This theoretical result is illustrated numerically.

Keywords: thermo-electro-elastic material, frictional contact, finite element method, error estimate, numerical simulations

1. Introduction

Piezoelectric materials are used as distributed sensors and actuators in many engineering applications because of their direct and converse piezoelectric effects. Industrial piezoelectric devices are subject to high temperatures, thus must be designed to withstand thermal effects. It has also been recognized that thermally induced deformation-stress is an essential consideration in the distributed sensing and control of laminated structures with integrated piezoelectric actuators or sensors. Thus, a coupling of thermo-electro-mechanical fields is needed to be taken into account if a temperature load is considered in a piezoelectric solid. Several models of fully coupled thermo-piezoelectricity has been developed for determining static responses under combined thermal, electric and mechanical excitations, see (Liu *et al.*, 2014; Shang *et al.*, 2002; Sládek *et al.*, 2010; Tiersten, 1971). Recently, contact problems involving thermo-piezoelectric materials (Baiz *et al.*, 2018; Benaissa *et al.*, 2015, 2016) have been studied.

The present article is concerned with the numerical modeling of unilateral contact problems in a thermo-electro-elastic material with the Tresca friction law and regularized electrical and thermal conductivity conditions. The current paper is devoted to the numerical solution of the contact model introduced in (Benaissa *et al.*, 2015), and extends the results of (Essoufi *et al.*, 2015) to the case of thermo-electro-elastic materials. In the present paper, we use the boundary conditions on the contact surface used in (Essoufi *et al.*, 2015) for electro-elastic materials, which take into account the electric conductivity of the foundation. But, unlike (Essoufi *et al.*, 2015), in this work, we study, from the numerical point of view, a frictional contact problem between a thermo-electro-elastic body and an electrically and thermally conductive foundation.

The main novelty of this model lies in the chosen thermo-electro-elastic behavior for the body and in the electrical and thermal conditions describing the contact. The considered model leads to a new and more interesting mathematical model, involving new operators and new functionals. The analysis and numerical approach of this system represent the main trait of novelty of the present paper. To this end, we consider a discrete scheme to approximate the problem, based on the finite element method. We treat the friction unilateral contact by using a penalty method approach and a version of Newton's method. We implement this scheme in a numerical code and present numerical simulations in the study of a two-dimensional test problem.

The paper is organized as follows. In Section 2, we present a brief description of the mechanical model and its variational formulation. Details on the spatial discretization of the variational formulation using the finite element method are given in Section 3. A main error estimates result is proved, Theorem 3.1, from which the linear convergence of the algorithm is deduced under suitable regularity conditions. The numerical algorithm used for solving the discrete problem is described in Section 4, where some numerical examples are also presented in order to demonstrate the accuracy and the performance of the method. Finally, in Section 5 we present some conclusions and perspectives.

2. Problem statement

Consider a body made of a thermo-electro-elastic material which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$. The body is submitted to the action of body forces of density $\mathbf{f}_0$, a volume electric charges of density ϕ_0 and a heat source of constant strength ϑ_0. It is also submitted to mechanical, electric and thermal constraints on the boundary. To describe them, we consider a partition of Γ into three measurable parts Γ_1, Γ_2, Γ_3, on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas\Gamma_1 > 0$ and $meas\Gamma_a > 0$. The body is clamped on Γ_1, therefore, the displacement field vanishes there. Moreover, we assume that density of traction forces, denoted by $\mathbf{f}_2$, acts on the boundary part Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface electric charge of density ϕ_2 is prescribed on Γ_b. We suppose that temperature vanishes in $\Gamma_1 \cup \Gamma_2$. Over the contact surface Γ_3, the body comes in frictional contact with a conductive foundation.

We denote by $\mathbb{S}^d$ the space of the second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of the order d. Also, below $\boldsymbol{\nu} = \{\nu_i\}$ represents the unit outward normal on Γ while " $\cdot$ " and $\|\cdot\|$ denotes the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively, that is $\mathbf{u}\cdot\mathbf{v} = u_i v_i$, $\|\mathbf{v}\| = (\mathbf{v}\cdot\mathbf{v})^{1/2}$ for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, and $\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$, $\|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2}$ for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d$. Here and everywhere in this paper i, j, k, l run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, i.e. $f_{,i} = \partial f/\partial x_i$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, by $u_\nu = \mathbf{u}\cdot\boldsymbol{\nu}$, $\mathbf{u}_\tau = \mathbf{u} - u_\nu\boldsymbol{\nu}$, $\sigma_\nu = \sigma_{ij}\nu_i\nu_j$, and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$.

The classical model for the process is as in the following.

Problem P. *Find a displacement field $\mathbf{u} : \Omega \to \mathbb{R}^d$, an electric potential field $\varphi : \Omega \to \mathbb{R}$ and a temperature field $\theta : \Omega \to \mathbb{R}$ such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{E}^*\mathbf{E}(\varphi) - \theta\mathcal{M} \quad \text{in} \quad \Omega \tag{2.1}$$

$$\mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{\eta}\mathbf{E}(\varphi) - \theta\mathcal{P} \quad \text{in} \quad \Omega \tag{2.2}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in} \quad \Omega \tag{2.3}$$

$$\operatorname{div}\mathbf{D} = \phi_0 \quad \text{in} \quad \Omega \tag{2.4}$$

$$\operatorname{div}\mathbf{q} = \vartheta_0 \quad \text{in} \quad \Omega \tag{2.5}$$

$$\mathbf{q} = -\mathcal{K}\nabla\theta \quad \text{in} \quad \Omega \tag{2.6}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_1 \tag{2.7}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{f}_2 \quad \text{on} \quad \Gamma_2 \tag{2.8}$$

$$\sigma_\nu \leqslant 0, \quad u_\nu - g \leqslant 0, \quad \sigma_\nu(u_\nu - g) = 0 \qquad \text{on} \quad \Gamma_3 \tag{2.9}$$

$$\|\boldsymbol{\sigma}_\tau\| \leqslant S, \quad \boldsymbol{\sigma}_\tau = -S\frac{\mathbf{u}_\tau}{\|\mathbf{u}_\tau\|} \quad \text{if} \quad \mathbf{u}_\tau \neq \mathbf{0} \quad \text{on} \quad \Gamma_3 \tag{2.10}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a \tag{2.11}$$

$$\mathbf{D}\cdot\boldsymbol{\nu} = \phi_2 \quad \text{on} \quad \Gamma_b \tag{2.12}$$

$$\mathbf{D}\cdot\boldsymbol{\nu} = p_e(u_\nu - g)\phi_L(\varphi - \varphi_f) \quad \text{on} \quad \Gamma_3 \tag{2.13}$$

$$\theta = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2 \tag{2.14}$$

$$\mathbf{q}\cdot\boldsymbol{\nu} = p_c(u_\nu - g)\phi_L(\theta - \theta_f) \quad \text{on} \quad \Gamma_3 \tag{2.15}$$

In (2.1)-(2.15) and below, in order to simplify the notation, we do not indicate the dependence of the functions on the spatial variable $\mathbf{x} \in \Omega \cup \Gamma$.

Equations (2.1) and (2.2) represent the thermo-electro-elastic constitutive law of the material in which $\boldsymbol{\sigma} = (\sigma_{ij})$ denotes the stress tensor, $\boldsymbol{\varepsilon}(\mathbf{u}) = (\boldsymbol{\varepsilon}_{ij}(\mathbf{u}))$ denotes the linearized strain tensor, $\mathbf{E}(\varphi)$ is the electric field. $\mathcal{F} = \{f_{ijkl}\}$, $\mathcal{E} = \{e_{ijk}\}$, $\boldsymbol{\eta} = (\beta_{ij})$, $\mathcal{M} = (m_{ij})$ and $\mathcal{P} = \{p_i\}$ are respectively, the elasticity, piezoelectric, electric permittivity, thermal expansion and pyroelectric tensors. $\mathcal{E}^*$ is the transpose of $\mathcal{E}$. We recall that $\boldsymbol{\varepsilon}_{ij}(\mathbf{u}) = (u_{i,j} + u_{j,i})/2$ and $\mathbf{E}(\varphi) = -\nabla\varphi = -\{\varphi_{,i}\}$. Also the tensors $\mathcal{E}$ and $\mathcal{E}^*$ satisfy the equality

$$\mathcal{E}\boldsymbol{\sigma}\cdot\mathbf{v} = \boldsymbol{\sigma}\cdot\mathcal{E}^*\mathbf{v} \qquad \forall \boldsymbol{\sigma} \in \mathbb{S}^d, \quad \mathbf{v} \in \mathbb{R}^d$$

and the components of the tensor $\mathcal{E}^*$ are given by $e^*_{ijk} = e_{kij}$. Equations (2.3)-(2.5) represent the equilibrium equations for the stress, electric displacement fields and heat flux vector, respectively, in which Div and div denote the divergence operators for the tensor and vector valued functions, i.e. $\operatorname{Div}\boldsymbol{\sigma} = \{\sigma_{ij,j}\}$, $\operatorname{div}\mathbf{D} = \{D_{i,i}\}$, (2.6) is the Fourier law of heat conduction with $\mathcal{K} = \{k_{ij}\}$ denoting the thermal conductivity tensor. Relations (2.7)-(2.8), (2.11)-(2.12) and (2.14)) represent the mechanical, electric and thermal boundary conditions. Unilateral boundary conditions (2.9) represent the Signorini law, in which g is the gap in the reference configuration between Γ_3 and the foundation, measured along the direction of $\boldsymbol{\nu}$ and (2.10) represents the Tresca friction law, in which S is the given slip bound on Γ_3.

Equation (2.15) represents the regularisation thermal contact condition on Γ_3, where $p_c:\ r \to p_c(r)$ is the thermal conductance function, supposed to be zero for $r < 0$ and positive otherwise, and θ_f is the foundation temperature. Relation (2.13) represents regularization of the electrical contact condition on Γ_3, similar to that used in (Barboteu and Sofonea, 2009), where φ_f represents the electric potential of the foundation. Finally, the truncation function ϕ_L is defined by $\phi_L(s) = s$ if $|s| \leqslant L$ and $\phi_L(s) = (s/|s|)L$ if $|s| > L$, where L is a large positive constant.

We now turn to the variational formulation of Problem P which is the starting point for the numerical modelling based on the finite element discretization. To this end, we use the notation $H = [L^2(\Omega)]^d$ and we introduce the spaces

$$V = \{\mathbf{v} \in [H^1(\Omega)]^d;\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\} \qquad W = \{\xi \in H^1(\Omega);\ \xi = 0 \text{ on } \Gamma_a\}$$
$$Q = \{\eta \in H^1(\Omega);\ \eta = 0 \text{ on } \Gamma_1 \cup \Gamma_2\} \qquad \mathcal{H} = \{\boldsymbol{\tau} = (\tau_{ij});\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\}$$

The spaces H, V, W, Q and $\mathcal{H}$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u},\mathbf{v})_H = \int\limits_\Omega \mathbf{u}\cdot\mathbf{v}\,d\mathbf{x} \qquad (\mathbf{u},\mathbf{v})_V = \int\limits_\Omega \boldsymbol{\varepsilon}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{v})\,d\mathbf{x}$$
$$(\varphi,\xi)_W = \int\limits_\Omega \nabla\varphi\cdot\nabla\xi\,d\mathbf{x} \qquad (\theta,\eta)_Q = \int\limits_\Omega \nabla\theta\cdot\nabla\eta\,d\mathbf{x}$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int\limits_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,d\mathbf{x}$$

We introduce the convex cone K of admissible displacements which satisfy the noninterpenetration on the contact zone Γ_3

$$K = \{\mathbf{v} \in V;\ v_\nu - g \leqslant 0 \quad \text{on} \quad \Gamma_3\}$$

The following assumptions about the problem data will be needed later.

($\mathbf{h}_1$) The tensor $\mathcal{F} :\ \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ and the tensors $\boldsymbol{\eta}, \mathcal{K} :\ \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the usual properties of symmetry, boundedness, and ellipticity

$$f_{ijkl} = f_{klij} = f_{jikl} \in L^\infty(\Omega) \qquad \beta_{ij} = \beta_{ji} \in L^\infty(\Omega) \qquad k_{ij} = k_{ji} \in L^\infty(\Omega)$$

and there exist positive constants $m_{\mathcal{F}}$, $m_{\boldsymbol{\eta}}$, and $m_{\mathcal{K}}$ such that

$$\begin{aligned} &f_{ijkl}(\mathbf{x})\tau_{ij}\tau_{kl} \geqslant m_{\mathcal{F}}\|\boldsymbol{\tau}\|^2 \qquad \forall \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d \qquad \forall \mathbf{x} \in \Omega \\ &\beta_{ij}(\mathbf{x})E_iE_j \geqslant m_{\boldsymbol{\eta}}\|\mathbf{E}\|^2 \qquad k_{ij}(\mathbf{x})E_iE_j \geqslant m_{\mathcal{K}}\|\mathbf{E}\|^2 \qquad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d \qquad \forall \mathbf{x} \in \Omega \end{aligned}$$

($\mathbf{h}_2$) The tensors $\mathcal{E} :\ \Omega \times \mathbb{S}^d \to \mathbb{S}^d$, $\mathcal{M} :\ \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ and $\mathcal{P} :\ \Omega \times \mathbb{R} \to \mathbb{R}^d$ satisfy the following properties

$$e_{ijk} = e_{ikj} \in L^\infty(\Omega) \qquad m_{ij} = m_{ji} \in L^\infty(\Omega) \qquad p_i \in L^\infty(\Omega)$$

($\mathbf{h}_3$) The surface electrical conductivity $p_e :\ \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ and the thermal conductance $p_c :\ \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfy for $\pi = p_e$ or p_c:

$$\begin{aligned} &\exists M_\pi > 0 \quad \text{such that} \quad |\pi(x,u)| \leqslant M_\pi \quad \forall u \in \mathbb{R}\ \wedge\ x \in \Gamma_3 \\ &x \to \pi(x,u) \quad \text{is measurable on} \quad \Gamma_3 \quad \forall u \in \mathbb{R} \\ &\pi(x,u) = 0 \qquad \forall x \in \Gamma_3\ \wedge\ u \leqslant 0 \end{aligned}$$

($\mathbf{h}_4$) The functions $u \to \pi(x,u)$ for $\pi = p_e$ (resp. p_c) are Lipschitz functions on $\mathbb{R}$ for all $x \in \Gamma_3$:

$$|\pi(x,u_1) - \pi(x,u_2)| \leqslant L_\pi|u_1 - u_2| \qquad \forall u_1, u_2 \in \mathbb{R} \quad \text{with} \quad L_\pi > 0$$

($\mathbf{h}_5$) The given forces, charge densities and heat source satisfy

$$\begin{aligned} &\mathbf{f}_0 \in L^2(\Omega)^d \qquad \mathbf{f}_2 \in L^2(\Gamma_2)^d \qquad \phi_0 \in L^2(\Omega) \\ &\phi_2 \in L^2(\Gamma_b) \qquad \vartheta_0 \in L^2(\Omega) \end{aligned}$$

($\mathbf{h}_6$) The potential and temperature of the contact surface satisfy

$$\varphi_f \in L^2(\Gamma_3) \qquad \theta_f \in L^2(\Gamma_3)$$

($\mathbf{h}_7$) The gap function and the friction bound satisfy

$$g \in L^2(\Gamma_3) \quad g \geqslant 0 \qquad S \in L^\infty(\Gamma_3) \quad S \geqslant 0$$

We consider the functionals $j :\ V \to \mathbb{R}_+$, $l :\ V \times W \times W \to \mathbb{R}$ and $\chi :\ V \times Q \times Q \to \mathbb{R}$ defined by

$$\begin{aligned} &j(\mathbf{v}) = \int_{\Gamma_3} S|\mathbf{v}_\tau|\,da \qquad \forall \mathbf{v} \in V \\ &l(\mathbf{u},\varphi,\xi) = \int_{\Gamma_3} p_e(u_\nu - g)\phi_L(\varphi - \varphi_f)\xi\,da \qquad \forall \mathbf{u} \in V \quad \forall \varphi, \xi \in W \\ &\chi(\mathbf{u},\theta,\eta) = \int_{\Gamma_3} p_c(u_\nu - g)\phi_L(\theta - \theta_f)\eta\,da \qquad \forall \mathbf{u} \in V \quad \forall \theta, \eta \in Q \end{aligned}$$

Using the Riesz theorem, we define the linear mappings $\mathbf{f} \in V$, $\phi \in W$ and $\vartheta \in Q$ as follows

$$(\mathbf{f}, \mathbf{v})_V = \int_\Omega \mathbf{f}_0 \cdot \mathbf{v}\, d\mathbf{x} + \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v}\, da \qquad \forall \mathbf{v} \in V$$

$$(\phi, \xi)_W = \int_\Omega \phi_0 \xi\, d\mathbf{x} - \int_{\Gamma_b} \phi_2 \xi\, da \qquad \forall \xi \in W$$

$$(\vartheta, \eta)_Q = \int_\Omega \vartheta_0 \eta\, d\mathbf{x} \qquad \forall \eta \in Q$$

Proceeding in a standard way, we obtain the following variational formulation of Problem P.

Problem P_V. *Find a displacement field $\mathbf{u} \in K$, an electric potential field $\varphi \in W$ and a temperature field $\theta \in Q$ such that*

$$\begin{aligned}
&\big(\mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})\big)_{\mathcal{H}} + \big(\mathcal{E}^*\nabla\varphi, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})\big)_{\mathcal{H}} - \big(\mathcal{M}\theta, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})\big)_{\mathcal{H}} \\
&\qquad + j(\mathbf{v}) - j(\mathbf{u}) \geqslant (\mathbf{f}, \mathbf{v} - \mathbf{u})_V \qquad \forall \mathbf{v} \in K \\
&(\boldsymbol{\eta}\nabla\varphi, \nabla\xi)_H - \big(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}), \nabla\xi\big)_H - (\mathcal{P}\theta, \nabla\xi)_H + l(\mathbf{u}, \varphi, \xi) = (\phi, \xi)_W \qquad \forall \xi \in W \\
&(\mathcal{K}\nabla\theta, \nabla\eta)_H + \chi(\mathbf{u}, \theta, \eta) = (\vartheta, \eta)_Q \qquad \forall \eta \in Q
\end{aligned}$$

We complete this Section with a result on existence and uniqueness of solution to Problem P_V.

Theorem 2.1. *Assume* $(\mathbf{h}_1)$-$(\mathbf{h}_3)$ *and* $(\mathbf{h}_5)$-$(\mathbf{h}_7)$ *hold. Then one has the following:*

(1) The Problem P_V has at least one solution.

(2) Under $(\mathbf{h}_4)$*, there exists a constant $L^* > 0$ such that if* $M_{p_e} + M_{p_c} + LL_{p_e} + LL_{p_c} + \max(\|\mathcal{M}\|, \|\mathcal{P}\|) < L^*$*, then the problem P_V has a unique solution.*

Here the norms of the tensors $\mathcal{P} = \{p_i\}$ and $\mathcal{M} = \{m_{ij}\}$ are given by $\|\mathcal{P}\| = \max_{1 \leqslant i \leqslant d} \|p_i\|$, $\|\mathcal{M}\| = \max_{1 \leqslant i,j \leqslant d} \|m_{ij}\|$.

The proof for Theorem 2.1 is given in (Benaissa *et al.*, 2015).

3. Numerical analysis

This Section is devoted to numerical discretization of the Problem P_V. First, we consider three finite dimensional spaces $V^h \subset V$, $W^h \subset W$ and $Q^h \subset Q$ approximating the spaces V, W and Q, respectively, in which $h > 0$ denotes the spatial discretization parameter. In addition, we consider the discrete set of admissible displacements defined by $K^h = K \cap V^h$. In the numerical simulations presented in the next Section, V^h, W^h and Q^h consist of continuous and piecewise affine functions, that is

$$\begin{aligned}
V^h &= \{\mathbf{v}^h \in [C(\overline{\Omega})]^d; \quad \mathbf{v}^h_{|Tr} \in [P_1(Tr)]^d, \quad Tr \in \mathcal{T}^h, \quad \mathbf{v}^h = \mathbf{0} \ \text{ on } \ \Gamma_1\} \\
W^h &= \{\xi^h \in C(\overline{\Omega}); \quad \xi^h_{|Tr} \in P_1(Tr), \quad Tr \in \mathcal{T}^h, \quad \xi^h = 0 \ \text{ on } \ \Gamma_a\} \\
Q^h &= \{\eta^h \in C(\overline{\Omega}); \quad \eta^h_{|Tr} \in P_1(Tr), \quad Tr \in \mathcal{T}^h, \quad \eta^h = 0 \ \text{ on } \ \Gamma_1 \cup \Gamma_2\}
\end{aligned}$$

where Ω is assumed to be a polygonal domain, $\mathcal{T}^h$ denotes finite element triangulation of $\overline{\Omega}$, and $P_1(Tr)$ represents space of polynomials of a global degree less or equal to one in Tr.

The finite element approximation of the Problem P_V is the following.

Problem P_V^h. *Find a discrete displacement field $\mathbf{u}^h \in K^h$, a discrete electric potential field $\varphi^h \in W^h$ and a discrete temperature field $\theta^h \in Q^h$ such that*

$$
\begin{aligned}
&\big(\mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}^h), \boldsymbol{\varepsilon}(\mathbf{v}^h) - \boldsymbol{\varepsilon}(\mathbf{u}^h)\big)_{\mathcal{H}} + \big(\mathcal{E}^*\nabla\varphi^h, \boldsymbol{\varepsilon}(\mathbf{v}^h) - \boldsymbol{\varepsilon}(\mathbf{u}^h)\big)_{\mathcal{H}} - \big(\mathcal{M}\theta^h, \boldsymbol{\varepsilon}(\mathbf{v}^h) - \boldsymbol{\varepsilon}(\mathbf{u}^h)\big)_{\mathcal{H}} \\
&\qquad + j(\mathbf{v}^h) - j(\mathbf{u}^h) \geqslant (\mathbf{f}, \mathbf{v}^h - \mathbf{u}^h)_V \qquad \forall \mathbf{v}^h \in K^h \\
&(\boldsymbol{\eta}\nabla\varphi^h, \nabla\xi^h)_H - \big(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}^h), \nabla\xi^h\big)_H - (\mathcal{P}\theta^h, \nabla\xi^h)_H + l(\mathbf{u}^h, \varphi^h, \xi^h) \\
&\qquad = (\phi, \xi^h)_W \qquad \forall \xi^h \in W^h \\
&(\mathcal{K}\nabla\theta^h, \nabla\eta^h)_H + \chi(\mathbf{u}^h, \theta^h, \eta^h) = (\vartheta, \eta^h)_Q \qquad \forall \eta^h \in Q^h
\end{aligned}
$$

On the assumptions of Theorem 2.1, the discrete problem P_V^h has a unique solution. We now focus on the error analysis between the solutions to Problems P_V and P_V^h. Our main result in this matter is the following.

Theorem 3.1. *Assume the conditions of Theorem 2.1 hold. Then there exists a constant c, independent of h, such that*

$$
\begin{aligned}
&\|\mathbf{u} - \mathbf{u}^h\|_V + \|\varphi - \varphi^h\|_W + \|\theta - \theta^h\|_Q \\
&\qquad \leqslant c \inf_{\substack{\mathbf{v}^h \in U^h \\ \xi^h \in W^h \\ \eta^h \in Q^h}} \Big\{ \|\mathbf{u} - \mathbf{v}^h\|_V + \|\mathbf{u} - \mathbf{v}^h\|_V^{\frac{1}{2}} + \|\varphi - \xi^h\|_W + \|\theta - \eta^h\|_Q \Big\}
\end{aligned}
\tag{3.1}
$$

The proof of Theorem 3.1 is done by using properties ($\mathbf{h}_1$)-($\mathbf{h}_7$), and applying several times the inequality $ab \leqslant \delta a^2 + (1/4\delta)b^2$, $a, b, \delta \in \mathbb{R}$ for some $\delta > 0$ small enough, and after some tedious algebraic manipulations. Inequality (3.1) is a basis for deriving error estimation and convergence analysis. In an analogous way, if we also suppose that $\boldsymbol{\sigma\nu} \in L^2(\Gamma_3)^d$ and using the classical results of interpolation (see Ciarlet, 1978), we have the following result.

Corollary 3.1. Assume the hypothesis of Theorem 3.1 and, in addition, assume that $\mathbf{u} \in H^2(\Omega)^d$, $\mathbf{u}_{|\Gamma_3} \in H^2(\Gamma_3)^d$, $\varphi \in H^2(\Omega)$, $\theta \in H^2(\Omega)$ and $\boldsymbol{\sigma\nu} \in L^2(\Gamma_3)^d$. Then there exists a constant c, independent of h, such that

$$
\|\mathbf{u} - \mathbf{u}^h\|_V + \|\varphi - \varphi^h\|_W + \|\theta - \theta^h\|_Q \leqslant ch \tag{3.2}
$$

4. Numerical results

In this Section, we present first the numerical scheme which we have implemented. Then, we describe a two-dimensional example of the numerical results, which we obtained by employing it to show the performance of the method.

4.1. Numerical scheme

Let N_{tot}^h be the total number of nodes and denote by α^i, β^i, γ^i the basis functions of the spaces V^h, W^h and Q^h, respectively, for $i = 1, \ldots, N_{tot}^h$. Then, the expression of functions $\mathbf{v}^h \in V^h$, $\xi^h \in W^h$ and $\eta^h \in Q^h$ is given by

$$
\mathbf{v}^h = \sum_{i=1}^{N_{tot}^h} \mathbf{v}^i \alpha^i \qquad \xi^h = \sum_{i=1}^{N_{tot}^h} \xi^i \beta^i \qquad \eta^h = \sum_{i=1}^{N_{tot}^h} \eta^i \gamma^i
$$

where $\mathbf{v}^i$, ξ^i and η^i represent the values of the corresponding functions $\mathbf{v}^h$, ξ^h and η^h at the i-th node of $\mathcal{T}^h$.

The penalized approach we use shows that the Problem P_V^h can be governed by the system of nonlinear equations

$$\mathbf{R}(\mathbf{u},\varphi,\theta)=\mathbf{G}(\mathbf{u},\varphi,\theta)+\mathbf{F}(\mathbf{u},\varphi,\theta)=\mathbf{0} \tag{4.1}$$

where the functions $\mathbf{G}$ and $\mathbf{F}$ are defined below. Here, the vectors $\mathbf{u}\in\mathbb{R}^{d\times N_{tot}^h}$, $\varphi\in\mathbb{R}^{N_{tot}^h}$ and $\theta\in\mathbb{R}^{N_{tot}^h}$ represent respectively the generalized vectors defined as follows

$$\mathbf{u}=\{\mathbf{u}^i\}_{i=1}^{N_{tot}^h}\qquad \varphi=\{\varphi^i\}_{i=1}^{N_{tot}^h}\qquad \theta=\{\theta^i\}_{i=1}^{N_{tot}^h}$$

where $\mathbf{u}^i$, φ^i, and θ^i represent the value of the function $\mathbf{u}^h$, φ^h and θ^h at the i-th nodes of $\mathcal{T}^h$.

The thermo-electro-elastic term $\mathbf{G}(\mathbf{u},\varphi,\theta)\in\mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}$ is defined by

$$\begin{aligned}
&\big(\mathbf{G}(\mathbf{u},\varphi,\theta)\cdot(\mathbf{v},\xi,\eta)\big)_{\mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}}\times\mathbb{R}^{N_{tot}}}=\big(\mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}^h)-\mathcal{M}\theta^h,\boldsymbol{\varepsilon}(\mathbf{v}^h)\big)_{\mathcal{H}}\\
&\qquad+\big(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{v}^h),\nabla\varphi^h\big)_H-\big(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}^h)-\boldsymbol{\eta}\nabla\varphi^h+\mathcal{P}\theta^h,\nabla\xi^h\big)_H\\
&\qquad+(\mathcal{K}\nabla\theta^h,\nabla\eta^h)_H-(\mathbf{f},\mathbf{v}^h)_V-(\phi,\xi^h)_W-(\vartheta,\eta^h)_Q\\
&\forall\mathbf{v}\in\mathbb{R}^{d\times N_{tot}^h}\qquad \xi\in\mathbb{R}^{N_{tot}^h}\qquad \eta\in\mathbb{R}^{N_{tot}^h}\qquad \forall\mathbf{v}^h\in V^h\\
&\xi^h\in W^h\qquad \eta^h\in Q^h
\end{aligned}$$

Above, $\mathbf{v}$, ξ and η represent the generalized vectors of components $\mathbf{v}^i$, ξ^i and η^i, for $i=1,\dots,N_{tot}^h$, respectively, and note that the volume and surface efforts are contained in the term $\mathbf{G}\mathbf{u},\varphi,\theta)$.

Finally, the specific penalized contact operator $\mathbf{F}(\mathbf{u},\varphi,\theta)\in\mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}$, which permits one to take into account the thermal and electrical conductivity of the foundation, is given by

$$\begin{aligned}
&\big(\mathbf{F}(\mathbf{u},\varphi,\theta)\cdot(\mathbf{v},\xi,\eta)\big)_{\mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}}=\int\limits_{\Gamma_3}c_\nu(u_\nu^h-g^h)_+v_\nu^h\,da\\
&\qquad+\int\limits_{\Gamma_3}P_{B(0,S)}(c_\nu\mathbf{u}_\tau^h)\cdot\mathbf{v}_\tau^h\,da+l(\mathbf{u}^h,\varphi^h,\xi^h)+\chi(\mathbf{u}^h,\theta^h,\eta^h)\\
&\forall\mathbf{v}\in\mathbb{R}^{d\times N_{tot}^h}\qquad \xi\in\mathbb{R}^{N_{tot}^h}\qquad \eta\in\mathbb{R}^{N_{tot}^h}\\
&\forall\mathbf{v}^h\in V^h\qquad \xi^h\in W^h\qquad \eta^h\in Q^h
\end{aligned}$$

where c_ν is a positive penalty coefficient, $P_{B(0,S)}$ is the orthogonal projection on the closed ball of center 0 and radius S, with S Tresca's threshold, and x_+ denotes the positive part of $x\in\mathbb{R}$, i.e. $x_+=\max\{0,x\}$.

A Newton type algorithm is used to solve problem (4.1); this solution permits one to treat, at the same time, both the triple $(\mathbf{u},\varphi,\theta)$ that we denote by the variable $\mathbf{x}$ thereafter. This Newton algorithm can be summarized by the following iteration process

$$\begin{aligned}
&\mathbf{x}^{i+1}=\mathbf{x}^i-(\mathbf{K}^i+\mathbf{T}^i)^{-1}\big(\mathbf{G}(\mathbf{u}^i,\varphi^i,\theta^i)+\mathbf{F}(\mathbf{u}^i,\varphi^i,\theta^i)\big)\\
&\mathbf{K}^i=D_{\mathbf{u},\varphi,\theta}\mathbf{G}(\mathbf{u}^i,\varphi^i,\theta^i)\qquad \mathbf{T}^i=D_{\mathbf{u},\varphi,\theta}\mathbf{F}(\mathbf{u}^i,\varphi^i,\theta^i)
\end{aligned}$$

where $\mathbf{x}^{i+1}$ denotes the triple $(\mathbf{u}^{i+1},\varphi^{i+1},\theta^{i+1})$ and i represents the Newton iteration index. Here, $D_{\mathbf{u},\varphi,\theta}\mathbf{G}$ and $D_{\mathbf{u},\varphi,\theta}\mathbf{F}$ denote differentials of the functions $\mathbf{G}$ and $\mathbf{F}$ with respect to the variables $\mathbf{u}$, φ and θ. This leads us to solve the resulting linear system

$$(\mathbf{K}^i+\mathbf{T}^i)\Delta\mathbf{x}^i=-\mathbf{G}(\mathbf{u}^i,\varphi^i,\theta^i)-\mathbf{F}(\mathbf{u}^i,\varphi^i,\theta^i)$$

where $\Delta\mathbf{x}^i = (\Delta\mathbf{u}^i, \Delta\varphi^i, \Delta\theta^i)$ with $\Delta\mathbf{u}^i = \mathbf{u}^{i+1} - \mathbf{u}^i$, $\Delta\varphi^i = \varphi^{i+1} - \varphi^i$ and $\Delta\theta^i = \theta^{i+1} - \theta^i$. For more considerations about Computational Contact Mechanics, see the recent monograph (Alart and Curnier, 1991; Laursen, 2002; Renard, 2013; Wriggers, 2002).

Note that formulation (4.1) has been implemented in our open-source finite element library GetFEM++ (see http: //getfem.org/).

4.2. Numerical simulations

We now want to verify the linear convergence of the numerical scheme proposed. In order to do that, we consider the physical problem depicted in Fig. 1, where a rectangular body is in contact with a conductive foundation. The domain $\varOmega = (0,4) \times (0,1)$ is a cross section of a three-dimensional rectangular body clamped on $\varGamma_1 = (\{0\} \times [0,1]) \cup (\{4\} \times [0,1])$ and the electric potential is free there (we choose $\varGamma_1 = \varGamma_a$). Let $\varGamma_2 = \varGamma_b = ([0,1] \times \{0\}) \cup ([3,4] \times \{0\}) \cup ([0,4] \times \{1\})$. The body is subjected to action of surface tractions acting on $[1,3] \times \{1\}$, i.e. $\mathbf{f}_2 = (0, -10^2)\,\mathrm{N/m^2}$, while the remainder of the part $\varGamma_2$ is free, and no electric charges are applied in the surface. We suppose that the temperature vanishes in $\varGamma_1 \cup \varGamma_2$. The body is in contact with a foundation on $\varGamma_3 = [1,3] \times \{0\}$. No volume forces, no electric charges and no volume heat source are supposed to act in the body, i.e. $\mathbf{f}_0 = \mathbf{0}\,\mathrm{N/m^3}$, $\phi_0 = 0\,\mathrm{C/m^3}$, $\vartheta_0 = 0\,\mathrm{W/m^3}$.

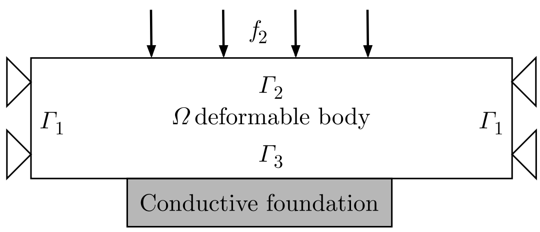

Fig. 1. Contact problem with a conductive foundation

The truncation function ϕ_L and the conductivity functions p_r $(r = e, c)$ in conditions (2.13) and (2.15) are given by

$$\phi_L(s) = s \qquad p_r(s) = k_r \cdot \begin{cases} 0 & \text{if} \quad s < 0 \\ \dfrac{s}{\epsilon_r} & \text{if} \quad 0 \leqslant s \leqslant \epsilon_r \\ 1 & \text{if} \quad s > \epsilon_r \end{cases}$$

where k_r and ϵ_r $(r = e, c)$ are positive constants. The rest of the data are the following: $c_\nu = 10^7\,\mathrm{N/m^2}$, $g = 0\,\mathrm{m}$, $S = 8.5\,\mathrm{N/m^2}$, $\epsilon_e = 10^{-6}$, $k_e = 0.1$, $\varphi_f = -44\,\mathrm{V}$, $\epsilon_c = 10^{-6}$, $k_c = 1$, $\theta_f = 373\,\mathrm{K}$.

Thee material parameters of $\mathrm{B_aTiO_3}$ are taken as (Liu *et al.*, 2014):

- Elastic [GPa]: $f_{11} = 166$, $f_{13} = 78$, $f_{33} = 162$, $f_{44} = 43$;
- Piezoelectric [$\mathrm{C/m^2}$]: $e_{31} = -4.4$, $e_{33} = 18.6$, $e_{15} = 11.6$;
- Dielectric [C/GVm]: $\beta_{11} = 11.2$, $\beta_{33} = 12.6$;
- Thermal expansion [$\times 10^6\,\mathrm{N/Km^2}$]: $m_{11} = 2.24$, $m_{33} = 1.89$;
- Pyroelectric [$\times 10^{-4}\mathrm{C/Km^2}$]: $p_1 = 0$, $p_3 = -1$;
- Heat conduction coefficients [W/Km]: $k_{11} = 50$, $k_{33} = 75$.

In the plane of deformations setting, constitutive equations (2.1)-(2.2) can be written by using a compressed matrix notation in place of the tensor notation as follows

$$\begin{bmatrix}\boldsymbol{\sigma}_{11}\\ \boldsymbol{\sigma}_{33}\\ \boldsymbol{\sigma}_{13}\\ \mathbf{D}_1\\ \mathbf{D}_3\end{bmatrix}=\begin{bmatrix} f_{11} & f_{13} & 0 & 0 & e_{31}\\ f_{13} & f_{33} & 0 & 0 & e_{33}\\ 0 & 0 & f_{44} & e_{15} & 0\\ 0 & 0 & e_{15} & -\beta_{11} & 0\\ e_{31} & e_{33} & 0 & 0 & -\beta_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{33}\\ 2\varepsilon_{13}\\ -\mathbf{E}_1\\ -\mathbf{E}_3\end{bmatrix}+\begin{bmatrix}-m_{11}\\ -m_{33}\\ 0\\ -p_1\\ -p_3\end{bmatrix}$$

First, the deformed mesh and the initial boundary are plotted in Fig. 2a. The values of the elastic constraints and the computed electric potential in the deformed configuration are presented in Figs. 2b and 3a, respectively. It follows from these figures that the inverse piezoelectric effect is respected, i.e. the appearance of mechanical stress in the body due to the action of the electric field. Also, the simulations underline the effects of the electrical conductivity of the foundation on the process. In Fig. 3b, the temperature field is plotted on the deformed configuration. Clearly, effects due to the influence of foundation temperature, can be observed.

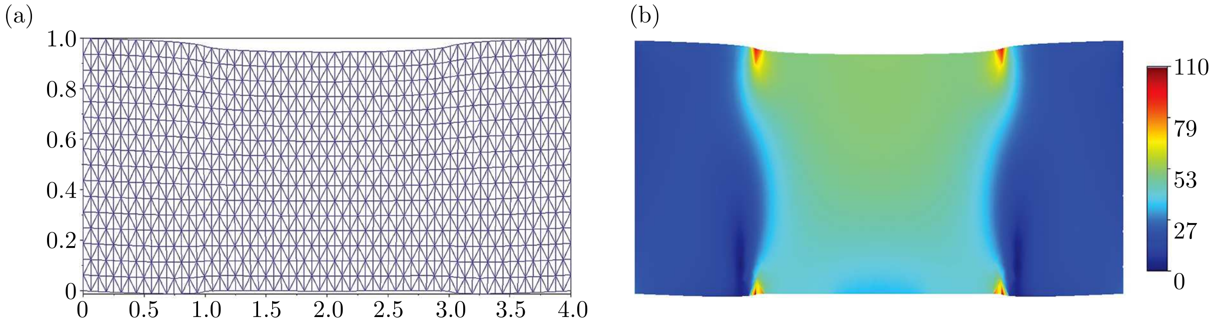

Fig. 2. Deformed mesh (a) and elastic constraints in the deformed configuration (b)

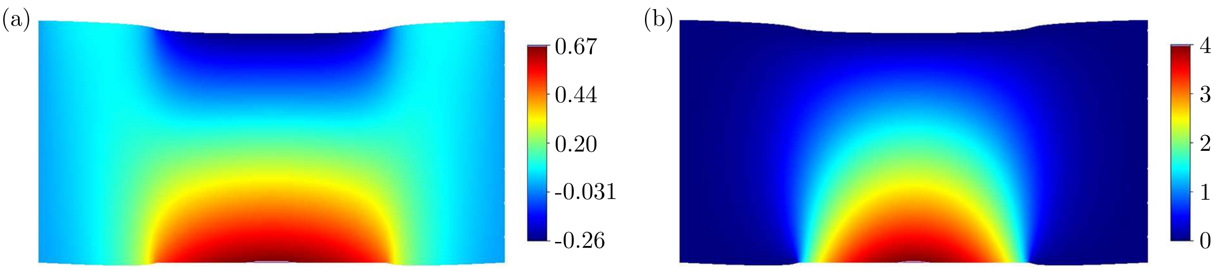

Fig. 3. Electric potential (a) and the temperature field (b) in the deformed configuration

To see the convergence behaviour of the discrete scheme, we compute a sequence of numerical solutions based on uniform triangulations of the domain $[0,4]\times[0,1]$. Then, we provide the estimated error values for several discretization parameters h in the form (see Corollary 3.1)

$$E^h=\|\mathbf{u}-\mathbf{u}^h\|_V+\|\varphi-\varphi^h\|_W+\|\theta-\theta^h\|_Q$$

Here the sides of the rectangle are divided into $1/h$ equal parts. We start with $h=1/2$ which are successively halved. The numerical solution corresponding to $h=1/64$ is taken as the "exact" solution, which is used to compute the errors of numerical solutions with larger values of h; this finer discretization corresponds to a problem with around 67723 degrees of freedom. The numerical results are presented in Fig. 4 where the dependence of the error estimate E^h with respect to h is plotted. The curve of the numerical error estimate is asymptotically linear, which is consistent with the theoretically predicted optimal linear convergence of the numerical solution established in (3.2).

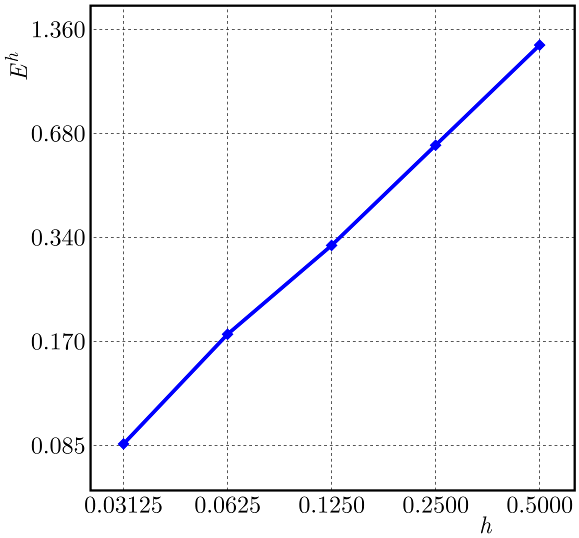

Fig. 4. Estimated errors

5. Conclusion

A new model of the contact process between a thermo-piezoelectric body and the foundation is numerically studied in this paper. The novelties arise in the fact that the material is assumed to be thermo-electro-elastic and the foundation is thermally-electrically conductive. A discrete scheme was used to approach the problem and an optimal order error estimate was derived. A numerical algorithm which combined the penalty approach with the Newton method was implemented. Moreover, numerical simulations for a representative two-dimensional example were provided. These simulations validate the theoretical error estimates and, in addition, allow one to study the influence of electric potential and temperature field of the foundation on the process. The algorithm may be used as a benchmark for calibration of computer codes for more complicated thermo-piezoelectric contact problems. This work opens the way to study further problems with other conditions for thermally-electrically conductive taking into the account frictional heating effects, in a quasistatic case.

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